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Introduction of a Simple Cnoidal Wave Formulation Based on Nonlinear Interaction of Wave-Wave Principles

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ABSTRACT

In this study, a simple and efficient approach based on nonlinear wave interaction fundamentals is theoretically proposed to generate surface profile of the cnoidal waves. The approach includes Newton-Raphson algorithm to calculate the Ursell parameter and using a simple formulation to attain time series of cnoidal waves profile. The wave profile is determined as a superposition of limited number of cosine harmonics without encountering difficulties of using elliptic or hyperbolic functions, or any complex and complicated differential equations. It is demonstrated that a cnoidal wave profile is a result of high order self nonlinear interactions of primary frequency. Some definite energy is transmitted to higher harmonics due to nonlinear interactions. The amount of transmitted energy is controlled by Ursell parameter. The desirable accuracy determines the number of included harmonics in the proposed formulation and relative error of approach can be predicted based on Fourier and least square analysis techniques. The outputs of the proposed method are verified with cnoidal resulted from elliptic functions and the high efficiency of new approximation is revealed for engineering applications. The calculation of wave parameters such as energy flux, volume flux and radiation stress for cnoidal wave can be approximated using the proposed method. Using this approach, a physical interpretation of the B_m parameter (introduced in the first order of cnoidal wave theory) is presented. The calculation of several parameters such as velocity vectors and dynamic pressure of cnoidal waves is very simple by means of proposed approach.

1. Introduction

As waves propagate toward shallow waters, the sea bed disturbs the orbital motions of particles and asymmetrical surface level profiles are appeared. Some of shallow water phenomena such as shoaling, breaking and triad interactions become more significant gradually and affect the wave profile and energy spectrum, potentially. Some solutions such as higher order Stokes wave and perturbation theories are understood to explain the nonlinear waves' characteristics in the shallow waters. As well as, cnoidal wave theory can be implemented to describe the nonlinear waves in the shallow waters.

The triad interactions can be introduced as the most important responsible for waves' nonlinearity which is usually active before wave breaking in the shoaling zone (1). Phillips (2) showed theoretically that the second-order Stokes wave is an outcome of a nonlinear interaction of two primary wave trains. Longuet-Higgins and Smith (3) confirmed this theory by means of laboratory observations.

The energy spectra of shoaling waves show growth in secondary or even tertiary peaks as the result of nonlinear interactions. The generation and amplification of these harmonics were studied and investigated by several observational and numerical studies (4-8). Results showed that these harmonics are bound with the spectral peak (8-11). The second and third peaks of energy spectrum on two and three times of spectral peak frequency are very usual for field measurements (12, 13). The nonlinearity of waves can be explained as the energy transition to higher or lower harmonics bounded with main frequency (14). This restrictive condition becomes disappeared or insignificant for breaker or de-shoaling waves.

The triad interaction is an energy exchange among three waves satisfying the following relationships (3, 6, 8, 15)

$$f_1 \pm f_2 = f_3 \quad (1)$$
$$k_1 \pm k_2 = k_3 \quad (2)$$

where f_i and k_i are scalar frequency and wave-number of *i*th component of an irregular wave train resulted from Fourier analysis, respectively. Both primary components satisfy the linear dispersion relationship (Eq. 3) but the third component might not. The liner dispersion relation is

$$\omega^2 = gk \tanh kd \quad (3)$$

where g is the gravitational acceleration and $\omega = 2\pi f$. When third component also satisfies the linear dispersion relationship, resonance occurs and considerable energy is transferred to this component. In a resonant interaction, the amplitude of the third component can grow as large as those of the primary components. In non-resonant or so called bound interactions, the change in the amplitude of consequent component is small and this type of interaction happens for non-dispersive waves in shallow waters (2).

In a train of wave with a distinct spectral peak f_p containing considerable energy, the self-interaction of spectral peak component $(f_1 = f_2 = f_p)$ is more feasible than any other combination of primary frequencies (9, 16). Some studies highlighted that the significant triad interactions occurred between f_p and $2f_p$ (1, 4). In some other studies higher harmonics up to $4f_p$ were investigated (14).

As mentioned, cnoidal wave theory can be utilized to explain and determine the distorted linear profile and nonlinear properties of asymmetrical waves in shallow water. This theory presents the cnoidal wave profile as function of elliptic parameters as Eq. (4)

$$\eta(t,x) = H_{cn}\left[\frac{1}{m}\left(1-\frac{E}{K}\right)-1+cn^2\left(2K\left(\frac{t}{T_{cn}}-\frac{x}{L_{cn}}\right),m\right)\right] \quad (4)$$

where H_{cn}, T_{cn} and L_{cn} are height, period and length of cnoidal wave, respectively. Also, *cn* is a jacobi elliptic function and *m*, *E* and *K* are elliptic parameters which their relatively complicated definitions are out of the scope of present study.

Solution of cnoidal wave theory was presented by Korteweg and De Vries (17). Second order of cnoidal wave theory was presented by Laitone (18). The fifth order solution of cnoidal wave height was introduced by Monkmeyer (19) and improved by Fenton (20), and later by Rienecker and Fenton (21). Some of studies tried to simplify the available formulas without deteriorating their performances (e.g. Fenton and Gardiner-Garden (22)). If cnoidal waves are assumed as a result of wave by wave nonlinear interactions, it seems that simple introduction of this type of wave could be presented. This clue is the motivation of the present study.

In the present study, it is attempted to generate the first order cnoidal wave profile based on a simple formulation and the fact that the self-spectral peak triad interaction in higher orders is the sole significant interaction. Therefore, the energy of integer multiples of spectral peak frequencies grows as the result of higher order interactions. The maximum error of approach is clarified by combination of least square and Fourier analysis. The novelty of this study is presenting an approach for engineering applications which has appropriate efficiency and applicability without encountering complicated differential equations, hyperbolic and elliptic functions.

2. Methodology

Spectral analysis is founded by Fourier transform. Most of spectral characteristics of wave train data are based on the energy density variance spectrum which is resulted by spectral analysis. The Fourier analysis with almost no limitation is a useful and suitable technique to determine the exact energy content and its distribution on indeterminate frequencies and harmonics. The number of harmonics attained by Fourier analysis is one half of discrete analyzed data amount.

Least square technique is another analysis method to distinct the contribution of several harmonics energy. The significant difference between this method and Fourier analysis is that the number and frequencies of involved harmonics in the least square analysis are limited and definite. Also, if the difference frequency of two selected consecutive harmonics is taken lesser, longer time series will be necessary to distinct the energy of harmonics and avoid ill condition for numerical matrix solution in this method. Therefore, the confidence of time series length is an important fact in the least square method. Otherwise, the resulted values of harmonic energy in ill condition are vague, not reliable and might be several ten times larger than the actual values.

In this study, the initial time series of first order cnoidal waves are generated using the Mike21 toolbox. The time series are set very dense with time step of 0.05 s to cover very high and steep waves. On the other hand, the Ursell parameter is calculated based on trinary cnoidal characteristics (height, wave length, depth) and using Newton-Rophson algorithm:

$$Ur = \frac{H_{cn} L_{cn}^2}{h^3} \quad (5)$$

where h is the water depth. The Ursell parameter is a non-dimensional number which demonstrates the degree of nonlinearity. Most of cnoidal wave parameters are controlled and governed by Ursell parameter (23).

Prepared time series resulted from Mike 21 toolbox are analyzed by least square method. The frequencies of harmonics are selected based on basic primary cnoidal frequency. In order to avoid uncertain results for higher harmonics, the time series were selected with sufficient length. Also, Fourier analysis is utilized to ensure the reliability of least square results. Since the time series of theoretical cnoidal wave have been assumed to be noiseless, therefore the length and segmentation of time series as well as increasing the analysis degree of freedom (d.o.f.) result insignificant changes in Fourier transform outcome (less than 0.05% in the worst attained result in the present study). The results of least square method will be only acceptable if no ill condition is attained in the matrix solution of least square method and the energy content of least square method doesn't exceed the Fourier ones. Otherwise, the time series extension or reduction of included harmonics is necessary and unavoidable. The energy content of each spectrum is proportional to summation of harmonic amplitude squares

$$E \propto \sum_{i=1}^{N} a_i^2 \qquad (6)$$

where, *a* stands for harmonic or constituent amplitudes in Fourier and least square methods, respectively. The relative error of least square analysis can be calculated by Eq. (7).

$$err = \frac{\left(\sum_{i=1}^{N_{hr}} a_i^2\right)_{Fr} - \left(\sum_{j=1}^{N_{cons}} a_j^2\right)_{ls}}{\left(\sum_{i=1}^{N_{hr}} a_i^2\right)_{Fr}} \times 100\% \qquad (7)$$

where N_{hr} and N_{cons} are number of harmonics and constituents in the Fourier and least square methods, respectively. The positive value of Eq. (7) is an indication of acceptable result of least square method.

The amplitudes of the harmonics resulted from least square method can be normalized by primary cnoidal height. The normalized amplitudes are called the amplitude coefficients (ACs) in this study.

3. Results and Discussions

3.1. New Approach Introduction

It is normally expected that the all of energy variance of an apparently linear wave is only attributed to the frequency of wave. In most cases, the outcomes are not consistent with this expectancy. It is usually to observe that considerable energy content is devoted to higher harmonics of Airy wave frequency. For instance, the amplitude spectrum of a cosine incident wave generated by a calibrated paddle in a two-dimensional flume is shown in Figure 1. The wave with height of 11 cm and period of 3 s was generated in 40 cm depth of water with control of paddle stroke according to Airy wave theory. The water fluctuations were recorded by a pressure transducer distanced 8.4 m far from paddle. Depth attenuation correction was accomplished on the pressure data. The reflected wave was removed using two other gauges and Mansard and Funke (24) method. In Figure 1, it can be observed that the multiples of main frequency dedicate considerable energy content (relating to amplitude square). As expected, the most energy proportion is pertaining to the main frequency and the proportion of energy content decreases with frequency increment. This evidence can be interpreted that the wave becomes nonlinear with the assumed characteristics in that water depth and all of bounded harmonics travel with an equal and certain celerity in a constant nonlinear wave profile. According to cnoidal wave theory, the Ursell parameter of this regular wave is equal to 63.

At first, it is supposed that cnoidal wave theory represent all characteristics of nonlinear regular waves. A first order cnoidal wave time series with characteristics of (H_{cn}=2.5m, T_{cn}=16s, depth=6.0m) is assumed and produced by conventional elliptic function (wave toolbox). The Ursell parameter of this wave is estimated as 212 using Newton-Rophson algorithm. The time series of intended wave was analyzed by both of Fourier and least square (based on 10 harmonics) methods. The Fourier analysis frequency resolution is governed by the length of time series. So, it is very usual that the energy of a certain intrinsic frequency may be distributed between two adjacent Fourier frequencies. However, the total energy resulted from Fourier analysis of several time series of a particular wave with different durations must be the same.

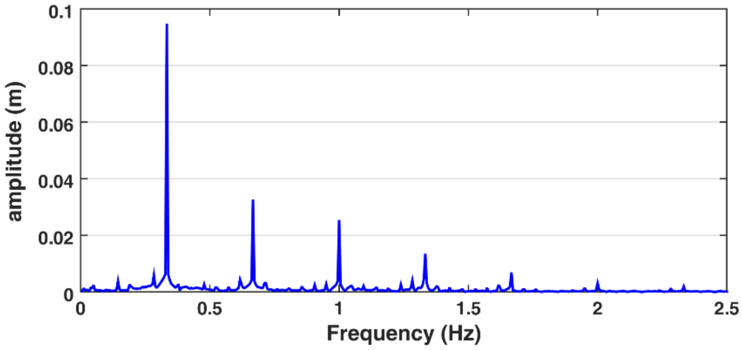

Figure 1. The amplitude spectrum for wave with (H=11cm, T=3s, depth=40cm)

Table 1. Results of least square analysis of cnoidal wave (Hcn=2.5m, Tcn=16s, depth=6.0m)

Con. No.	1	2	3	4	5	6	7	8	9	10
Period (s)	16.000	8.000	5.333	4.000	3.200	2.667	2.286	2.000	1.778	1.600
Amp (m)	0.717	0.542	0.359	0.218	0.124	0.068	0.036	0.019	0.010	0.005
Phase (deg)	44.830	89.661	134.491	179.322	224.152	268.983	313.813	358.644	43.475	88.305
AC	0.287	0.217	0.144	0.087	0.050	0.027	0.015	0.008	0.004	0.002

The result of least square analysis is somewhat different. The harmonic frequencies are defined and imposed by user (*n* times of main frequency in this case). Consequently, the exact energy content of each frequency is revealed. It is important to consider sufficient time duration of data to avoid the time length dependent results of energy (amplitude) for consecutive frequencies. The results of least square analysis of aforesaid wave are presented in Table 1.

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In table 1, it can be observed that the higher harmonic phase lags are integer multiple of the first harmonic $(\varphi_n = n\varphi_1)$. It means that all of harmonic are bound together and this evidence is exactly matched with the unique condition of wave by wave nonlinear interaction in higher orders and consistent with the key assumption of this study. The explained analysis and procedure is performed for about 400 different cnoidal waves with various trinary characteristics and Ursell parameters. These waves are selected in the range of weakly nonlinear to very steep waves. The breaking limit was ignored because the approach is only founded on theoretical and mathematical assumptions. Number of harmonics is selected as 25 in the least square analysis for all wave data. Finally, the variations of amplitude coefficients (ACs) for the first order of cnoidal wave versus Ursell parameter is illustrated in Figure 2.

It can be found that the values of ACs are independent of wave characteristics and only governed by Ursell parameter. Therefore, the formulation of time series of cnoidal wave profiles can be expressed as Eq. (8)

$$\eta(x,t) = \sum_{i=1}^{N} H_{cn}.AC_i \cos[i(\omega_{cn}t - \varphi_0)] \tag{8}$$

which ω_{cn} is the angular frequency of the cnoidal wave.

The remaining variable is N which can be determined according to the desirable accuracy.

Two different cnoidal waves were assumed to investigate the sensitivity of wave profile perfection to the number of included harmonics; a highly nonlinear cnoidal wave with bulk wave parameters of (H_{cn}=2.8m, T_{cn}=19s, depth=4.0m) and a weakly nonlinear wave with characteristics of (H_{cn}=2.0m, T_{cn}=11s, depth=4.0m). The Ursell numbers for above-mentioned waves were equal to 956 and 185, respectively. The values of ACs were tabulated in Table 2 for these waves.

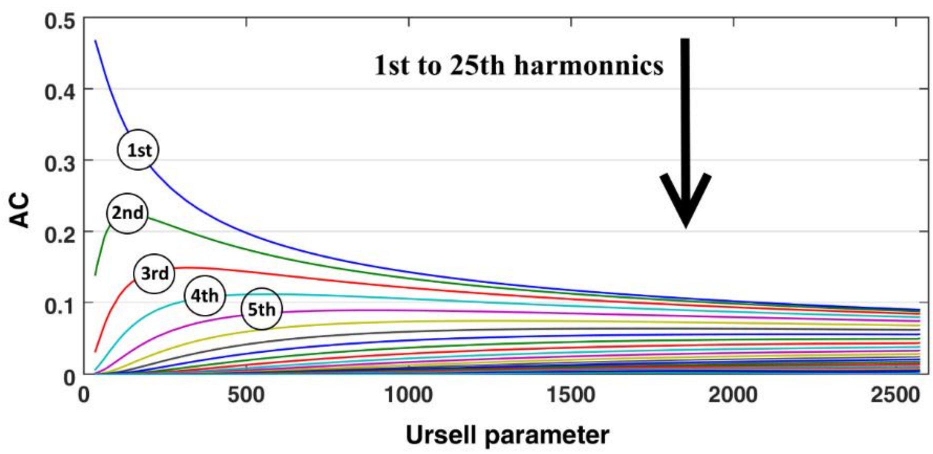

Figure 2. Variations of amplitude coefficients for the first order cnoidal wave for different harmonics

Table 2. The values of ACs for two different cnoidal waves; a) (Hcn=2.0m, Tcn=11s, depth=4.0m), b) (Hcn=2.8m, Tcn=19s, depth=4.0m)

Wave characteristics	Ur	1st *AC*	2nd *AC*	3rd *AC*	4th *AC*	5th *AC*	6th *AC*	7th *AC*	8th *AC*	9th *AC*	10th *AC*	11th *AC*	12th *AC*	13th *AC*	14th *AC*	15th *AC*
H_{cn}=2.0m, T_{cn}=11s, depth=4.0m	956	0.146	0.137	0.123	0.106	0.089	0.073	0.059	0.046	0.036	0.028	0.021	0.016	0.012	0.009	0.007
H_{cn}=2.8m, T_{cn}=19s, depth=4.0m	185	0.216	0.158	0.099	0.057	0.031	0.016	0.008	0.004	0.002	0.001	0	0	0	0	0

In Figures 3a and 3b, two cycles of three wave profiles adjusted to proposed approach with 3, 6 and 9 harmonics are demonstrated. Also, the primary time series of cnoidal waves initiated with Mike 21 toolbox were depicted in these diagrams. It can be realized that the noise was decreased and crests were become sharper and higher as more harmonics were included in the time series. With an equal number of included harmonics for two different cnoidal waves, the relative error is more significant for sharper wave (with superior nonlinearity degree and Ursell parameter).

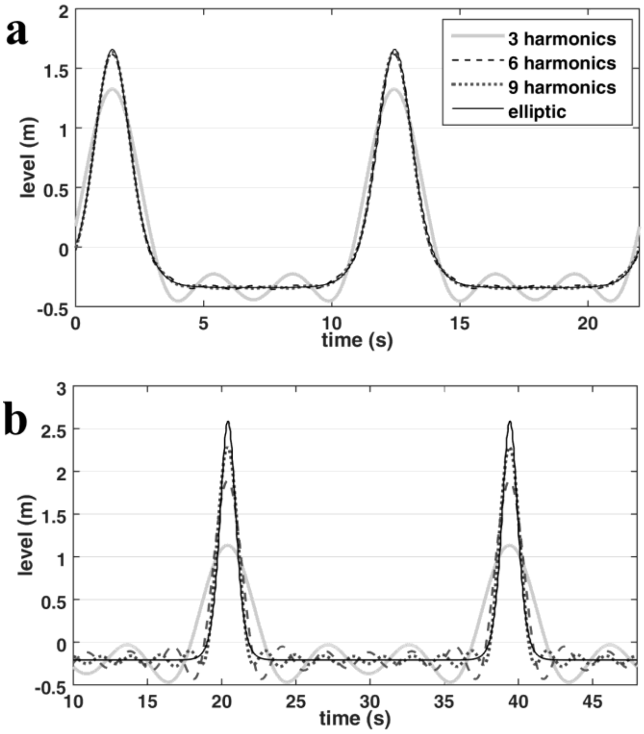

Figure 3. Sensitivity of wave profile perfection to the number of included harmonics; a) (Hcn=2.0m, Tcn=11s, depth=4.0m), b) (Hcn=2.8m, Tcn=19s, depth=4.0m)

The relative error of energy spectrum of Fourier and least square methods is calculated by Eq. (7). This value was computed for all of supposed cnoidal wave time series with Ursell parameter less than 1000. The variable *N* (number of harmonics) was varied from 1 to 15 and the result is illustrated in Figure 4. It is clear that more harmonics are necessary for minor error. On the other hand, for an equal accuracy, more harmonics are generally required for steeper waves with a larger Ursell parameters.

In Figure 4, it is resulted that for a permanent allowable energy error as 3%, two more harmonics are necessary to be included for 200 unit Ursell parameter increment. It is worth mentioning that 3% relative error of energy is pertaining to less than 2% of nominal cnoidal wave height error in the linear wave theory. This accuracy satisfies all field and laboratory coastal engineering applications.

If the wave length and cycle period of a regular wave is permanent, the wave profile will be stationary temporally and spatially. In this case, the celerity of wave is constant. We supposed that the cnoidal waves are result of some harmonic superposition. Therefore, this condition should be met by all of harmonics. This statement implies that the celerity of all harmonics is exact equal to original cnoidal wave. This means that these harmonics are bound together and not governed by dispersion relation of linear waves, which is also the indication of nonlinear wave interactions. The relation between frequency and wave number of each harmonic can be similarly written as cnoidal waves

$$\omega_i^2 = ghk_i^2\left[1+\frac{H}{mh}\left(2-m-3\frac{E}{K}\right)\right] \qquad (9)$$

in which g is the gravitational acceleration. The term of $\left(2-m-3\frac{E}{K}\right)$ might be expressed as a function of Ursell parameter (23).

Finally, the profile of water surface for cnoidal waves may be presented by the following Eq. (10):

$$\eta(x,t) = \sum_{i=1}^{N} H_{cn}.AC_i \cos\left[i(\omega_{cn}t - k_{cn}x - \varphi_0)\right] \qquad (10)$$

where k_{cn} is the cnoidal wave parameter. In this formulation, the proportions of the angular frequency (speed) and wave number of each harmonic to the main cnoidal wave parameters are equal. This means that the celerity of each harmonic is equal to the celerity of the original cnoidal wave. This is certainly a significance to retain the fixed and stationary wave profile.

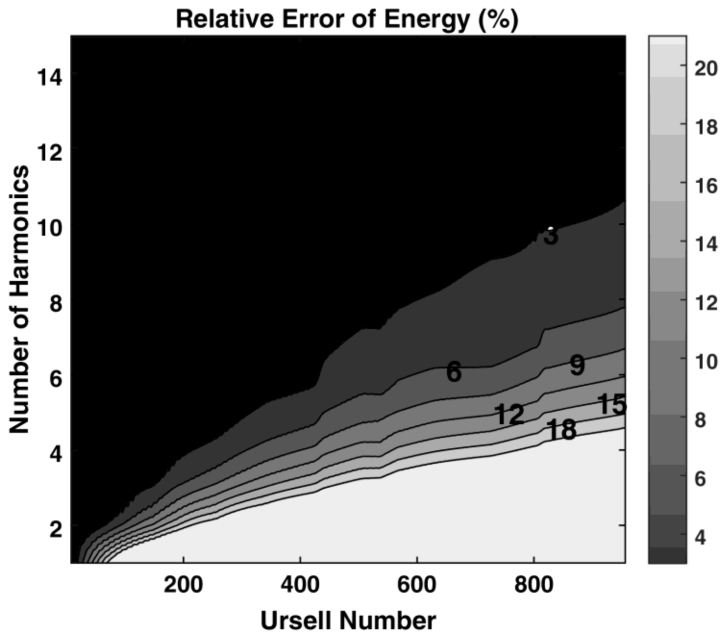

Figure 4. Number of necessary harmonics to satisfy the desirable accuracy

3.2. New Approach Efficiency Investigation

In the proposed approach, the surface profile of cnoidal wave has been replaced by summation of limited number of harmonics. Now it is attempted to assess the efficiency of this approach to estimate the physical parameters such as energy flux, volume flux and radiation stress. These parameters are calculated for each harmonic and added up to find out the same parameter for the cnoidal wave. According to the linear wave theory, the equations for energy flux, volume flux and radiation stress are as shown in the Eq. (11) to Eq. (13):

$$E_f = \frac{1}{16}\rho g H^2 c \ (1+G) \tag{11}$$

$$Q_w = \frac{1}{8}\frac{gH^2}{c} = \frac{1}{8}\frac{\omega H^2}{\tanh(kh)} \tag{12}$$

$$S_{xx} = \frac{1}{16}\rho g H^2 (1+2G) \tag{13}$$

in which H is the linear wave height, ρ is the water density, g is the gravitational acceleration and $G = \frac{2kh}{\sinh 2kh}$. Using proposed approach and considering shallow water approximations ($\tanh kh \sim kh$ and $G \sim 1$), the energy flux, volume flux and radiation stress could be written for cnoidal waves as Eq. (14) to Eq. (16), respectively.

$$E_f = \sum_{i=1}^{N}\frac{1}{16}\rho g H_i^2 c_{cn}(1+G) = \frac{1}{8}\rho g c_{cn}\sum_{i=1}^{N}(2H_{cn}.AC_i)^2 = 0.5\rho g H_{cn}^2 c_{cn}\sum_{i=1}^{N}AC_i^2 \tag{14}$$

$$Q_w = \sum_{i=1}^{N}\frac{1}{8}\omega\frac{H_i^2}{\tanh kh} = \sum_{i=1}^{N}\frac{1}{8}\omega\frac{H_i^2}{kh} = 0.5\frac{c_{cn}}{h}H_{cn}^2\sum_{i=1}^{N}AC_i^2 \tag{15}$$

$$S_{xx} = \sum_{i=1}^{N}\frac{1}{16}\rho g H_i^2 c_{cn}(1+2G) = \frac{3}{4}\rho g H_{cn}^2\sum_{i=1}^{N}AC_i^2 \tag{16}$$

The common term in these three statements is $\sum_{i=1}^{N}AC_i^2$.

This term can be calculated from Figure 2. This parameter is referred by SAC hereafter for shortening. The variation of this term versus Ursell parameter is plotted in Figure 5.

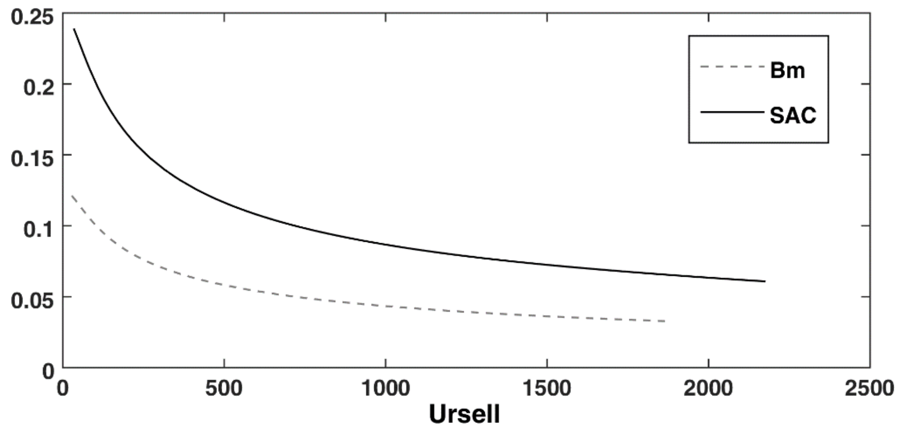

Figure 5. The variation of SAC and Bm versus Ursell number

Another interesting result is that the value of SAC is exactly twice times of Bm parameter defined by cnoidal wave theory (23). The value of Bm is calculated for cnoidal waves by

$$B(m)=\frac{1}{m^2}\left[\frac{1}{3}\left(3m^2-5m+2+(4m-2)\frac{E}{K}\right)-\left(1-m-\frac{E}{K}\right)^2\right] \quad (17)$$

This parameter varies from zero for very high and steep solitary waves up to 0.125 for linear waves. After replacing SAC by Bm, Eq. (18), (19) and (20) can be derived.

$$E_f=\rho g H_{cn}^2 c_{cn} B_m \quad (18)$$

$$Q_w=\frac{c_{cn}}{h}H_{cn}^2 B_m \quad (19)$$

$$S_{xx}=\frac{3}{2}\rho g H_{cn}^2 B_m \quad (20)$$

These formulas are exactly the same equations presented for the first order cnoidal wave theory (23). This outcome illustrates that the value of *Bm* is the half of SAC. It is quite obvious that the accuracy of resulted *Bm* and presented approach for calculating the above physical parameters is controlled by the number of included harmonics.

One of the significant advantages of this approach is the fact that it is simple to calculate *m*th order derivation of η versus *x*, as be derived as

$$\eta_{mx}=\sum_{i=1}^{N}(ik_{cn})^m.H_{cn}.AC_i.\sin\left(\theta+\frac{m\pi}{2}\right) \quad (21)$$

Replacing the general relationship of Eq. (21) in the velocities components (*u*,*v*) and the dynamic pressure equations resulted from cnoidal wave theory (23) (as Eq. (22) to Eq. (24)), returns the desirable parameters' values simply.

$$u=c\frac{\eta}{h}-c\left(\frac{\eta^2+\overline{\eta^2}}{h^2}\right)+\frac{1}{2}\cosh\left(\frac{1}{3}-\frac{(z+h)^2}{h^2}\right)\eta_{xx} \quad (22)$$

$$w=-c(z+h)\left[\frac{\eta_x}{h}\left(1+\frac{2\eta}{h}\right)+\frac{1}{6}h\left(1-\frac{(z+h)^2}{h^2}\right)\eta_{xxx}\right] \quad (23)$$

$$p_D=\rho g\left[\eta+\frac{1}{2}h^2\left(1-\frac{(z+h)^2}{h^2}\right)\eta_{xx}\right] \quad (24)$$

where

$$\overline{\eta^2}=\frac{Q_w.h}{c} \quad (25)$$

and *c* is the celerity of cnoidal wave and *z* is the parameter for vertical coordinate.

4. Conclusion

In this study, a simple formulation was proposed to generate the first order cnoidal wave profile. The basis of approach was founded on triad nonlinear interaction of wave by wave principles. Also, the Ursell number was calculated by Newton-Rophson algorithm. It was supposed that cnoidal wave profile was consequence of self-interaction of main frequency in high orders. Therefore, the total of cnoidal energy is distributed on the integer multiples of cnoidal frequencies. These harmonics are bound together with main frequency and propagate with the cnoidal celerity and the linear dispersion relation is not governing for these harmonics. So, it is normal that the wave length and period of each harmonic are the same factor of primary cnoidal wave's ones. On the other hand, the phase lags of these harmonics are also equal to inverse of same factor multiplied with cnoidal phase lag.

Mike 21 toolbox was implemented to generate the time series of several (~400) cnoidal waves with different characteristics and vast Ursell parameter range (up to 2500). Time series were analyzed by least square and Fourier methods to find the harmonics' oscillation amplitude and energy content of cnoidal wave.

It was revealed that the proportion of harmonics amplitude to the cnoidal wave height is only controlled by Ursell parameter. Finally, it was proposed that the cnoidal wave profile resulted from elliptic functions could be replaced by superposition of several cosine waves with certain and specified oscillation amplitude, frequency, wave number and phase lags. The only remaining parameter must be specified is the number of included harmonics in the superposition. The desirable accuracy and allowable error determine the number of harmonics. The relative error is introduced as the proportion of difference between energy contents resulted by Fourier and least square to total energy content resulted by means of Fourier analysis. For instance, eleven harmonics are necessary to include in the proposed formulation to avoid more than 3% energy error for a cnoidal wave with Ursell number of about 900. It was shown that in the range of Ursell number lesser than 1000, for 3% allowable error two more harmonics must be included for 200 unit Ursell parameter increment.

Finally, the proposed formulation performance was investigated by physical parameters such as energy and volume fluxes and radiation stress based on linear wave theory. It was demonstrated that the new approach is completely reliable and efficient. A physical definition of *Bm* (one of first order cnoidal parameter) is one of the secondary achievements of this study. It is found that this parameter is equal to the half of amplitude coefficient (proportion of harmonics amplitude to the cnoidal height) squares summation.

Another superiority of the proposed approach is the easiness of η_{mx} (the *m*th order derivation of water level to *x*) calculation. These parameters (η_{mx}) are implemented to calculate of several physical parameters such as velocity vectors and dynamic pressure of cnoidal waves.

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